ISING MODELS ON STATIC INHOMOGENEOUS RANDOM GRAPHS

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Abstract

On a finite inhomogeneous graph model for complex network, we define the Ising model, which is a paradigm model in statistical mechanics. For the ferromagnetic Ising model,we calculate the Thermodynamic limit of pressure per particle. From our results, we compute other physical quantities such as the magnetization and susceptibility, and investigate the critical behaviour of this model. Our calculations use large deviation principles (developed recently) for suitably defined empirical neighbourhood measures on inhomogeneous random graph.

1. Introduction

Statistical ferromagnetic models or ferromagnetic ising models on Power-law random graphs such as the configuration model (CM),have received an interestingly increasing attention in the last few years , see e.g. Montanari et al. [2010], Dommers et al. [2010] and the reference there in. For instance in Dembo and Montanari [2010] an explicit expression for the free-energy density and other important physical quantities such the *internal energy* in *thermodynamic limits* of $n \to \infty$, where n is number of sites on the graph, were obtained if the degree distribution of the model has finite mean.

Their analysis makes use of the local neighbourhood properties of a randomly chosen edge which looks like a homogeneous tree where the offspring have size-biased degree distribution. The Ising model on a tree is simplier to analyse, and many intense scientific research work on it has been taking place. This is because the effective field of a vertex can be expressed in terms of that of its neighbours via the distribution recursion. The relation between the CM and a tree allows for the calculation of the internal energy and upon integrating over all possible values of the inverse temperature parameter we obtain the pressure.

However, their technique cannot be applied to study the statistical mechanic on static inhomogeneous random graphs (SIRG) or coloured random graphs, see Penman [1998] or Cannings and Penman [2003] which have the Erdos-Renyi graphs as a special case, because it is not *locally-tree like* with an asymptotic distribution. Some research, see e.g Agliari et al. [2010], have use the Monte Carlo simulations to investigate the critical properties of the Ising model on Erdos-Renyi graphs.

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In this work we use the LDP techniques developed in Biggins and Penman [2003], Doku-Amponsah and Moeters [2010] or Doku-Amponsah [2006] and furthered in Doku-Amponsah [2011] to carry out Thermodynamic analysis of the Ising model defined on SIRG. To be specific, we obtain a thermodynamic limiting result of free-energy density via the random partition function. Using this results, we compute other physical quantities such as the, internal energy, the magnetization and the susceptibility of the system.

Our main motivation for looking at these models is the study of the two-populations Ising model. The two-populations Ising model is relevant in many phenomena, ranging from the study of anistropic magnetic materials to social economic models, to biological systems. In the case of the latter , an interesting application concerns cooperativity effects in the context of bacterial chemotaxis, where lasting puzzle, namely how a small change in extrenal concentration of attractant or repellent can cause significant amplification in receptor signal. In theory, models onthe two-population Ising model, although simplified, are able to show that a coupled system of receptors has the capacity to greatly amplify signals. These values in limit, approache the critical coupling energy of an analogous Ising models.See, e.g. Agliari et al. [2010] and the reference there in

2. Materials and Methods

2.1 The Ising model. Let V be a fixed set of n sites, say $V=\{1,\dots,n\}$ and $E\subset\mathcal{E}:=\big\{(u,v)\in V\times V\,:\,u<v\big\}$, where the *formal* ordering of links or bonds is introduced as a means to simply describe *unordered* bonds. Let $\mathcal{X}=\{-1,+1\}$ be the spin set and and denote by $\mathcal{G}_n(\mathcal{X})$ the set of all spinned graphs with spin set $\mathcal{X}$ and n sites.

Given a symmetric function $p_n\colon\mathbb{R}\times\mathbb{R}\to[0,1]$, a continuous function $s:\mathcal{X}\to\mathbb{R}$ and a probability measure ℓ on $s(\mathcal{X})$ we may define the *Inhomogeneous spinned random graph* or simply *spinned random graph* X with n sites as follows:

Assign to each site $v\in V$ *magnetic ising spin* $s(\eta(v))$ independently according to the *spin law* ℓ. Given the spins, we connect any two sites $u,v\in V$, independently of everything else, with a *bond probability* $p_n[s(\eta(u)),s(\eta(v))]$ otherwise keep them disconnected.

We always consider $X=\{\big(s(\eta(v))\,:\,v\in V\big),E\}$ under the joint law of graph and spin. We shall interpret X as spinned random graph and consider $s(\eta(v))$ as the spin of the site v.

On a spinned graph X, we define the ferromagnetic Ising Model by the following *Boltzmann distributions* over $\mathcal{X}^V$,

$$\mu_X(\eta)=\tfrac{1}{Z_X(\beta,B)}\exp\Big\{\sum_{(u,v)\in E}\tfrac{s(\eta(u))}{B(u)}\tfrac{s(\eta(v))}{B(v)}+\sum_{u\in V}\tfrac{s(\eta(u))}{\sqrt{\beta}}\Big\}$$

where $s(\eta(u))=\sqrt{\beta}B(\eta(u))\eta(u)$, $\beta\geq 0$ is the inverse temperature, $B=\{b(x):x\in\{-1,1\}^n\}$ is the vector external of magnetic fields, $\eta(u)\in\{-1,1\}$, and Z_X the random partition function is given by

$$Z_X(\beta,B):=\sum_{\eta\in\{-1,1\}^V}\exp\Big\{\sum_{(u,v)\in E}\tfrac{s(\eta(u))}{B(u)}\tfrac{s(\eta(v))}{B(v)}+\sum_{u\in V}\tfrac{s(\eta(u))}{\sqrt{\beta}}\Big\}$$

The free-energy density or the pressure per particle is defined by

$$\Phi_n(\beta,B)=\tfrac{1}{n}\log Z_n(\beta,B).$$

Our main concern in this paper is the study of the thermodynamic limiting behaviour of the free-energy. ie The behaviour of

$$\phi(\beta,B):=\lim_{n\to\infty}\Phi_n(\beta,B)$$

at different level of temperature. Throughout the rest of the paper we assume that, for large n, X is *sparse* or *near critical* case i.e. The bond probabilities satisfy $np_n[s(x),s(y)]\to C[s(x),s(y)]$, for all $x,y\in\{-1,1\}$ and $C\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$ is a continuous symmetric function, which is not identically equal to zero, and also satisfy the following *energetic preference* condition:

$$(e^{\beta}-1)\Big(C[-\sqrt{\beta}B(-1),-\sqrt{\beta}B(-1)]-C[\sqrt{\beta}B(1),\sqrt{\beta}B(1)]\Big)=2(B(-1)+B(1)). \tag{2.1}$$

2.2 Large deviation principles for spinned random graphs. In this subsection, we review some large deviation results of Doku et al. [2010] or Doku [2006], and extend the joint LDP for empirical measures of coloured random graph to spinned random graphs.i.e. we assume a more general spin law $\ell:\mathbb{R}\to[0,1]$ with all its exponential moments finite and prove an LDP for this model in a topology generated by the total variation norm.

To begin, we recall some useful definitions and notations from Doku et al. [2010] or Doku [2006]. A *rate function* is a non-constant, lower semicontinuous function I from a polish space $\mathcal{M}$ into $[0,\infty]$, it is called *good* if the level sets $\{I(m)\le x\}$ are compact for every $x\in[0,\infty)$. A functional M from the set of finite spinned graphs to $\mathcal{M}$ is said to satisfy a *large deviation principle* with rate function I if, for all Borel sets $B\subset\mathcal{M}$,

$$\begin{aligned}-\inf_{m\in\operatorname{int}B}I(m)&\le\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_n\big\{M(X)\in B\big\}\\&\le\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_n\big\{M(X)\in B\big\}\le-\inf_{m\in\operatorname{cl}B}I(m)\,,\end{aligned}$$

where X under $\mathbb{P}_n$ is a spinned random graph with n vertices and int B and cl B refer to the interior, resp. closure, of the set B.

And, for any finite or countable set $\mathcal{Y}$ we denote by $\mathcal{M}(\mathcal{Y})$ the space of probability measures, and by $\tilde{\mathcal{M}}(\mathcal{Y})$ the space of finite measures on $\mathcal{Y}$, both endowed with the topology generated by the total variation norm. For $\omega\in\tilde{\mathcal{M}}(\mathcal{Y})$ we denote by $\|\omega\|$ its total mass. Further, if $\ell\in\tilde{\mathcal{M}}(\mathcal{Y})$ and $\omega\ll\ell$ we denote by

$$H(\omega\,\|\,\ell)=\int_{\mathbb{R}}\omega[dy]\log\big(\tfrac{\omega[dy]}{\ell[dy]}\big)$$

the *relative entropy* of ω with respect to ℓ. We set $H(\omega\,\|\,\ell)=\infty$ if $\omega\not\ll\ell$. Finally, we denote by $\tilde{\mathcal{M}}_*(\mathcal{Y}\times\mathcal{Y})$ the subspace of symmetric measures in $\tilde{\mathcal{M}}(\mathcal{Y}\times\mathcal{Y})$.

On each spinned graph $X=((s(\eta(v))\,:\,v\in V),E)$ with n vertices, we define a probability measure, the *empirical spin measure* $L^1\in\mathcal{M}(\mathcal{X})$, by

$$L^1[x]:=\frac{1}{n}\sum_{v\in V}\delta_{s(\eta(v))}[x],\qquad\text{for }x\in\mathbb{R},$$

and a symmetric finite measure, the *empirical co-operative measure* $L^2\in\tilde{\mathcal{M}}_*(\mathbb{R}\times\mathbb{R})$, by

$$L^2[x,y]:=\frac{1}{n}\sum_{(u,v)\in E}(\delta_{(s(\eta(u),\,s(\eta(v))}+\delta_{(s(\eta(v),\,s(\eta(u))})[x,y],\qquad\text{for }a,b\in\mathcal{X}.$$

The total mass $\|L^2\|$ of the empirical pair measure is $2|E|/n$.

Theorem 2.1. *Suppose that X is a spinned random graph with spin law ℓ such that $n^{-1}\log\ell(n)\to-\infty$ and link probabilities $p_n\colon\mathbb{R}\times\mathbb{R}\to[0,1]$ satisfying $np_n[dx,dy]\to C[dx,dy]$, for all $x,y\in\mathbb{R}$ and some bounded symmetric measure $C\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$. Then, as $n\to\infty$, the pair (L^1,L^2) satisfies a large deviation principle in $\mathcal{M}(\mathbb{R})\times\tilde{\mathcal{M}}_*(\mathbb{R}\times\mathbb{R})$ with good rate function*

$$I(\omega,\varpi)=H(\omega\,\|\,\mu)+\frac{1}{2}\Big[H\big(\varpi\,\|\,C\omega\otimes\omega\big)+\|C\omega\otimes\omega\|-\|\varpi\|\Big]\,, \tag{2.2}$$

where the measure $C\omega\otimes\omega\in\tilde{\mathcal{M}}(\mathbb{R}\times\mathbb{R})$ is defined by $C\omega\otimes\omega[dx,dy]=C(x,y)\omega[dx]\omega[dy]$.

2.3 Exponential Change-of-Measure. Given a bounded function $\tilde{f}\colon\mathcal{X}\to\mathbb{R}$ and a symmetric bounded function $\tilde{g}\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}$, we define the constant $U_{\tilde{f}}$ by

$$U_{\tilde{f}}=\log\int_{\mathbb{R}}e^{\tilde{f}[x]}\ell[dx],$$

and the function $\tilde{h}_n\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$\tilde{h}_n[x,y]=\log\Big[\big(1-p_n[dx,dy]+p_n[dx,dy]e^{\tilde{g}[x,y]}\big)^{-n}\Big], \tag{2.3}$$

for $a,b\in\mathcal{X}$. We use $\tilde{f}$ and $\tilde{g}$ to define (for sufficiently large n) a new coloured random graph as follows:

- To the n labelled sites in V we assign spins from $s(\mathcal{X})$ independently and identically according to the spin law ℓ defined by
$$\tilde{\ell}[dx]=e^{\tilde{f}[x]-U_{\tilde{f}}}\ell[dx].$$
- Given any two sites $u,v\in V$, with u carrying spin a and v carrying spin b, we connect site u to site v with probability
$$\tilde{p}_n[dx,dy]=\frac{p_n[dx,dy]e^{\tilde{g}[x,y]}}{1-p_n[dx,dy]+p_n[dx,dy])e^{\tilde{g}[x,y]}}.$$

We denote the transformed law by $\tilde{\mathbb{P}}$. We observe that $\tilde{\ell}$ is a probability measure and that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$ as, for any spinned graph $X=((s(\eta(v))\colon v\in V),E)$,

$$\begin{aligned}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(X)&=\prod_{u\in V}\tfrac{\tilde{\ell}[ds(\eta(u)]}{\ell[ds(\eta(u)]}\prod_{(u,v)\in E}\tfrac{\tilde{p}_n[ds(\eta(u)),ds(\eta(v))]}{p_n[ds(\eta(u)),ds(\eta(v))]}\prod_{(u,v)\notin E}\tfrac{1-\tilde{p}_n[ds(\eta(u)),ds(\eta(v))]}{1-p_n[ds(\eta(u)),ds(\eta(v))]}\\&=\prod_{u\in V}\tfrac{\tilde{\ell}[ds(\eta(u)]}{\ell[ds(\eta(u)]}\prod_{(u,v)\in E}\tfrac{\tilde{p}_n[ds(\eta(u)),ds(\eta(v))]}{p_n[ds(\eta(u)),ds(\eta(v))]}\times\tfrac{n-np_n[ds(\eta(u)),ds(\eta(v))]}{n-n\tilde{p}_n[ds(\eta(u)),ds(\eta(v))]}\prod_{(u,v)\in\mathcal{E}}\tfrac{n-n\tilde{p}_n[ds(\eta(u)),ds(\eta(v))]}{n-np_n[ds(\eta(u)),ds(\eta(v))]}\\&=\prod_{u\in V}e^{\tilde{f}[s(\eta(u)]-U_{\tilde{f}}}\prod_{(u,v)\in E}e^{\tilde{g}[s(\eta(u)),s(\eta(v))]}\prod_{(u,v)\in\mathcal{E}}e^{\frac{1}{n}\tilde{h}_n[s(\eta(u)),s(\eta(v))]}\\&=\exp\big(n\langle L^1,\tilde{f}-U_{\tilde{f}}\rangle+n\langle\tfrac{1}{2}L^2,\tilde{g}\rangle+n\langle\tfrac{1}{2}L^1\otimes L^1,\tilde{h}_n\rangle-\langle\tfrac{1}{2}L_{\Delta}^1,\tilde{h}_n\rangle\big),\end{aligned} \tag{2.4}$$

where

$$L_{\Delta}^1=\tfrac{1}{n}\sum_{u\in V}\delta_{(X(u),X(u))}.$$

We write $\langle f,\omega\rangle:=\int_{\mathbb{R}}f[x]\omega[dx]$ and

$$\langle g,\varpi\rangle:=\int_{\mathbb{R}\times\mathbb{R}}g[x,y]\varpi[dx,dy].$$

Lemma 2.2 (Euler's lemma)**.** *If $np_n[dx,dy] \to C[dx,dy]$ for every $x,y \in \mathbb{R}$, then*

$$\lim_{n\to\infty}\Big(1+\alpha p_n[dx,dy]\Big)^n = e^{\alpha C[dx,dy]},\ \textit{for all } \alpha,x,y\in\mathbb{R} \textit{ and for } \alpha\in\mathbb{R}. \tag{2.5}$$

Proof. Observe that, for any $\varepsilon>0$ and for large n we have

$$\Big(1+\tfrac{\alpha C[dx,dy]-\varepsilon}{n}\Big)^n \le \Big(1+\alpha p_n[dx,dy]\Big)^n \le \Big(1+\tfrac{\alpha C[dx,dy]+\varepsilon}{n}\Big)^n,$$

by the pointwise convergence. Hence by the sandwich theorem and Euler's formula we get (2.5). ■

Lemma 2.3 (Exponential tightness)**.** *For every $\theta>0$ there exists $N\in\mathbb{N}$ such that*

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\Big\{|E|>nN\Big\}\le-\theta.$$

Proof. Let $c>\sup_{x,y\in\mathbb{R}}C[dx,dy]>0$. By a simple coupling argument we can define, for all sufficiently large n, a new coloured random graph $\tilde{X}$ with spin law ℓ and connection probability $\frac{c}{n}$, such that any edge present in X is also present in $\tilde{X}$. Let $|\tilde{E}|$ be the number of edges of $\tilde{X}$. Using Chebyshev's inequality, the binomial formula, and Lemma 2.2, we have that

$$\begin{aligned}\mathbb{P}\Big\{|\tilde{E}|\ge nl\Big\}\le e^{-nl}\mathbb{E}\big\{e^{|\tilde{E}|}\big\} &= e^{-nl}\sum_{k=0}^{\frac{n(n-1)}{2}} e^k\binom{n(n-1)/2}{k}\Big(\frac{c}{n}\Big)^k\Big(1-\frac{c}{n}\Big)^{n(n-1)/2-k}\\ &= e^{-nl}\Big(1-\frac{c}{n}+e\frac{c}{n}\Big)^{n(n-1)/2}\le e^{-nl}e^{nc(e-1+o(1))}.\end{aligned}$$

Now given $\theta>0$ choose $N\in\mathbb{N}$ such that $N>\theta+c(e-1)$ and observe that, for sufficiently large n,

$$\mathbb{P}\big\{|E|\ge nN\big\}\le\mathbb{P}\big\{|\tilde{E}|\ge nN\big\}\le e^{-n\theta},$$

which implies the statement. ■

Lemma 2.4 (Exponential tightness)**.** *For every $\theta>0$ there exists $K_\theta\subset\mathcal{M}(\mathbb{R})$ such that*

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\Big\{L^1\notin K_\theta\Big\}\le-\theta.$$

Proof. Let $1<l\in\mathbb{N}$ and choose $k(l)\in\mathbb{N}$, large enough, such that $\ell[e^{l^2\mathbb{1}_{\{s>k(l)\}}}]\le 2^l$
Then, using exponential Chebyschev's inequality, we have that

$$\mathbb{P}\Big\{\int_{\{x>k(l)\}}L^1(dx)\ge\tfrac{1}{l}\Big\}\le e^{-nl}\mathbb{E}\Big\{e^{l^2\sum_{u\in V}\mathbb{1}_{\{s(\eta(u))\ge k(l)\}}}\Big\}\le e^{-nl}\big(\ell[e^{l^2\mathbb{1}_{\{s>k(l)\}}}]\big)^n\le e^{-n(l-\log 2)}.$$

Now we fix $\theta>0$, choose $M>\theta+\log 2$, define the set Γ_M by

$$\Gamma_M:=\Big\{\omega:\int_{\{x\ge k(l)\}}\omega(dx)>\tfrac{1}{l},\ \text{for all } l\ge M\Big\}$$

As $\{x\le k(l)\}\subset\mathbb{R}$ is compact, the set Γ_M is pre-compact in the weak topology, by Prohorov's criterion. As

$$\mathbb{P}\Big\{L^1\notin\Gamma_M\Big\}\le\frac{1}{1-e^{-1}}\exp(-n[M-\log 2]),$$

we conclude that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big\{L^1\notin K_\theta\Big\}\le -\theta,$$

for the closure K_θ of Γ_M as required for the proof.

■

2.4 Proof of the upper bound in Theorem 3.4.

We denote by $\mathcal{C}_1$ the space of bounded functions on $\mathbb{R}$ and by $\mathcal{C}_2$ the space of bounded symmetric functions on $\mathbb{R}\times\mathbb{R}$, and define

$$\hat{I}(\omega,\varpi)=\sup_{\substack{f\in\mathcal{C}_1\\ g\in\mathcal{C}_2}}\Big\{\int_{\mathbb{R}}\big(f[x]-U_f\big)\omega[dx]+\tfrac{1}{2}\int_{\mathbb{R}\times\mathbb{R}}g[x,y]\varpi[dx,dy]+\tfrac{1}{2}\int_{\mathbb{R}\times\mathbb{R}}(1-e^{g[x,y]})C[dx,dy]\omega[dx]\omega[dy]\Big\}.$$

Lemma 2.5. *For each closed set $F\subset\mathcal{M}(\mathbb{R})\times\tilde{\mathcal{M}}_*(\mathbb{R}\times\mathbb{R})$,*

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\big\{(L^1,L^2)\in F\big\}\le-\inf_{(\omega,\varpi)\in F}\hat{I}(\omega,\varpi).$$

Proof. First let $\tilde{f}\in\mathcal{C}_1$ and $\tilde{g}\in\mathcal{C}_2$ be arbitrary. Define $\tilde{\beta}\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$\tilde{\beta}[x,y]=(1-e^{\tilde{g}[x,y]})C[dx,dy].$$

Observe that, by Lemma 2.2, $\tilde{\beta}[x,y]=\lim_{n\to\infty}\tilde{h}_n[x,y]$ for all $a,b\in\mathbb{R}$, recalling the definition of $\tilde{h}_n$ from (2.3). Hence, by (2.4), for sufficiently large n,

$$e^{\sup_{x\in\mathbb{R}}|\tilde{\beta}(x,x)|}\ge\int e^{\langle\frac{1}{2}L^1_\Delta,\,\tilde{h}_n\rangle}d\tilde{\mathbb{P}}=\mathbb{E}\Big\{e^{n\langle L^1,\tilde{f}-U_{\tilde{f}}\rangle+n\langle\frac{1}{2}L^2,\tilde{g}\rangle+n\langle\frac{1}{2}L^1\otimes L^1,\tilde{h}_n\rangle}\Big\},$$

where $L^1_\Delta=\frac{1}{n}\sum_{u\in V}\delta_{(X(u),X(u))}$ and therefore,

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}\Big\{e^{n\langle L^1,\tilde{f}-U_{\tilde{f}}\rangle+n\langle\frac{1}{2}L^2,\tilde{g}\rangle+n\langle\frac{1}{2}\,L^1\otimes L^1,\tilde{h}_n\rangle}\Big\}\le 0. \tag{2.6}$$

Given $\varepsilon>0$ let $\hat{I}_\varepsilon(\omega,\varpi)=\min\{\hat{I}(\omega,\varpi),\varepsilon^{-1}\}-\varepsilon$. Suppose that $(\omega,\varpi)\in F$ and observe that $\hat{I}(\omega,\varpi)>\hat{I}_\varepsilon(\omega,\varpi)$. We now fix $\tilde{f}\in\mathcal{C}_1$ and $\tilde{g}\in\mathcal{C}_2$ such that

$$\langle\tilde{f}-U_{\tilde{f}},\omega\rangle+\tfrac{1}{2}\,\langle\tilde{g},\varpi\rangle+\tfrac{1}{2}\,\langle\tilde{\beta},\omega\otimes\omega\rangle\ge\hat{I}_\varepsilon(\omega,\varpi).$$

As $\tilde{f},\tilde{g}$ are bounded functions, there exist open neighbourhoods B^2_ϖ and B^1_ω of ϖ,ω such that

$$\inf_{\substack{\tilde{\omega}\in B^1_\omega\\ \tilde{\varpi}\in B^2_\varpi}}\big\{\langle\tilde{f}-U_{\tilde{f}},\tilde{\omega}\rangle+\tfrac{1}{2}\,\langle\tilde{g},\tilde{\varpi}\rangle+\tfrac{1}{2}\,\langle\tilde{\beta},\tilde{\omega}\otimes\tilde{\omega}\rangle\big\}\ge\hat{I}_\varepsilon(\omega,\varpi)-\varepsilon.$$

Using Chebyshev's inequality and (2.6) we have that

$$\begin{aligned}&\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\big\{(L^1,L^2)\in B^1_\omega\times B^2_\varpi\big\}\\ &\quad\le\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}\Big\{e^{n\langle L^1,\tilde{f}-U_{\tilde{f}}\rangle+n\langle\frac{1}{2}L^2,\tilde{g}\rangle+n\langle\frac{1}{2}L^1\otimes L^1,\tilde{h}_n\rangle}\Big\}-\hat{I}_\varepsilon(\omega,\varpi)+\varepsilon\\ &\quad\le-\hat{I}_\varepsilon(\omega,\varpi)+\varepsilon.\end{aligned} \tag{2.7}$$

Now we use Lemma 2.3 with $\theta=\varepsilon^{-1}$, to choose $N(\varepsilon)\in\mathbb{N}$ and K_ε such that

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\Big\{|E|>nN(\varepsilon)\Big\}\le-\varepsilon^{-1}\ \text{ and }\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\Big\{L^1\notin K_\varepsilon\Big\}\le-\varepsilon^{-1} \tag{2.8}$$

For this $N(\varepsilon)$ and K_ε we define the set Γ_ε by

$$\Gamma_\varepsilon = \Big\{(\omega, \varpi) \in \mathcal{M}(\mathbb{R}) \times \tilde{\mathcal{M}}_*(\mathbb{R}\times\mathbb{R}) : \omega \in K_\varepsilon^c,\ \|\varpi\| \le 2N(\varepsilon)\Big\},$$

and recall that $\|L^2\| = 2|E|/n$. The set $\Gamma_\varepsilon \cap F$ is compact and therefore may be covered by finitely many sets $B^1_{\omega_r} \times B^2_{\varpi_r}, r = 1, \ldots, m$ with $(\omega_r, \varpi_r) \in F$ for $r = 1, \ldots, m$. Consequently,

$$\mathbb{P}\big\{(L^1, L^2) \in F\big\} \le \sum_{r=1}^{m} \mathbb{P}\big\{(L^1, L^2) \in B^1_{\omega_r} \times B^2_{\varpi_r}\big\} + \mathbb{P}\big\{(L^1, L^2) \notin \Gamma_\varepsilon\big\}.$$

We may now use (2.7) and (2.8) to obtain, for all sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}\big\{(L^1, L^2) \in F\big\} &\le \max_{r=1}^{m} \Big(\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}\big\{(L^1, L^2) \in B^1_{\omega_r} \times B^2_{\varpi_r}\big\}\Big) \vee (-\varepsilon)^{-1} \\ &\le \Big(- \inf_{(\omega,\varpi)\in F} \hat{I}_\varepsilon(\omega, \varpi) + \varepsilon\Big) \vee (-\varepsilon)^{-1}.\end{aligned}$$

Taking $\varepsilon \downarrow 0$ we get the desired statement. ■

Next, we express the rate function in term of relative entropies, see for example Dembo et al. [1998, (2.15)], and consequently show that it is a good rate function. Recall the definition of the function I from Theorem 3.4.

Lemma 2.6.

(i) *$\hat{I}(\omega, \varpi) = I(\omega, \varpi)$, for any $(\omega, \varpi) \in \mathcal{M}(\mathbb{R}) \times \tilde{\mathcal{M}}_*(\mathbb{R}\times\mathbb{R})$,*
(ii) *I is a good rate function and*
(iii) *$\mathfrak{H}_C(\varpi \,\|\, \omega) \ge 0$ with equality if and only if $\varpi = C\omega \otimes \omega$.*

Proof. (i) Suppose that $\varpi \not\ll C\omega \otimes \omega$. Then, there exists $a_0, b_0 \in \mathbb{R}$ with $C\omega \otimes \omega(a_0, b_0) = 0$ and $\varpi(a_0, b_0) > 0$. Define $\hat{g} \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$\hat{g}[x, y] = \log\big[K(\mathbb{1}_{(a_0,b_0)}[x, y] + \mathbb{1}_{(b_0,a_0)}[x, y]) + 1\big],\ \text{for } a, b \in \mathbb{R} \text{ and } K > 0.$$

For this choice of $\hat{g}$ and $f = 0$ we have

$$\begin{aligned}&\int_{\mathbb{R}} \big(f[x] - U_f\big)\omega(dx) + \int_{\mathbb{R}\times\mathbb{R}} \tfrac{1}{2}\hat{g}[x, y]\varpi[dx, dy] + \int_{\mathbb{R}\times\mathbb{R}} \tfrac{1}{2}(1 - e^{\hat{g}[x,y]})C[dx, dy]\omega[dx]\omega[dy] \\ &\ge \tfrac{1}{2}\log(K + 1)\varpi(a_0, b_0) \to \infty, \qquad \text{for } K \uparrow \infty.\end{aligned}$$

Now suppose that $\varpi \ll C\omega \otimes \omega$. We have

$$\begin{aligned}\hat{I}(\omega, \varpi) = &\sup_{f\in\mathcal{C}_1} \Big\{ \int_{\mathbb{R}} \Big(f[x] - \log \int_{\mathbb{R}} e^{f[x]}\ell[dx]\Big)\,\omega[dx]\Big\} \\ &+ \tfrac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} C[dx, dy]\omega[dx]\omega[dy] + \tfrac{1}{2}\sup_{g\in\mathcal{C}_2} \Big\{ \int_{\mathbb{R}\times\mathbb{R}} g[x, y]\varpi[dx, dy] - \int_{\mathbb{R}\times\mathbb{R}} e^{g[x,y]}C[dx, dy]\omega[dx]\omega[dy]\Big\}.\end{aligned}$$

By the variational characterization of relative entropy, the first term equals $H(\omega \,\|\, \ell)$. By the substitution $h = e^g \frac{C\omega\otimes\omega}{\varpi}$ the last term equals

$$\sup_{\substack{h\in\mathcal{C}_2\\ h\geq 0}} \int_{\mathbb{R}\times\mathbb{R}} \Big[\log\Big(h[x,y]\frac{\varpi[dx,dy]}{C[dx,dy]\omega[dx]\omega[dy]}\Big) - h[x,y]\Big]\,\varpi[dx,dy]$$

$$= \sup_{\substack{h\in\mathcal{C}_2\\ h\geq 0}} \sum_{a,b\in\mathbb{R}} \big(\log h[x,y] - h[x,y]\big)\,\varpi[dx,dy] + \sum_{a,b\in\mathbb{R}} \log\Big(\frac{\varpi[dx,dy]}{C[dx,dy]\omega[dx]\omega[dy]}\Big)\,\varpi[dx,dy]$$

$$= -\|\varpi\| + H(\varpi \,\|\, C\omega\otimes\omega),$$

where we have used $\sup_{x>0}\log x - x = -1$ in the last step. This yields that $\hat{I}(\omega,\varpi) = I(\omega,\varpi)$.

(ii) Recall from (2.2) that

$$I(\omega,\varpi) = H(\omega \,\|\, \ell) + \tfrac{1}{2}\,H\big(\varpi \,\|\, C\omega\otimes\omega\big) + \tfrac{1}{2}\,\|C\omega\otimes\omega\| - \tfrac{1}{2}\,\|\varpi\|.$$

All summands are continuous in ω,ϖ and thus I is a rate function. Moreover, for all $\alpha<\infty$, the level sets $\{I\leq\alpha\}$ are contained in the bounded set $\{(\omega,\varpi)\in\mathcal{M}(\mathbb{R})\times\tilde{\mathcal{M}}_*(\mathbb{R}\times\mathbb{R})\colon\ \mathfrak{H}_C(\varpi\,\|\,\omega)\leq\alpha\}$ and are therefore compact. Consequently, I is a good rate function.

(iii) Consider the nonnegative function $\xi(x) = x\log x - x + 1$, for $x>0$, $\xi(0)=1$, which has its only root in $x=1$. Note that

$$\mathfrak{H}_C(\varpi\,\|\,\omega) = \begin{cases} \int \xi\circ g\; dC\omega\otimes\omega & \text{if } g := \frac{d\varpi}{dC\omega\otimes\omega}\geq 0 \text{ exists,}\\ \infty & \text{otherwise.}\end{cases} \tag{2.9}$$

Hence $\mathfrak{H}_C(\varpi\,\|\,\omega)\geq 0$, and, if $\varpi = C\omega\otimes\omega$, then $\xi(\frac{d\varpi}{dC\omega\otimes\omega}) = \xi(1) = 0$ and so $\mathfrak{H}_C(C\omega\otimes\omega\,\|\,\omega) = 0$. Conversely, if $\mathfrak{H}_C(\varpi\,\|\,\omega) = 0$, then $\varpi[dx,dy]>0$ implies $C\omega\otimes\omega[x,y]>0$, which then implies $\xi\circ g[x,y] = 0$ and further $g[x,y]=1$. Hence $\varpi = C\omega\otimes\omega$, which completes the proof of (iii). ■

2.5 Proof of the lower bound in Theorem 3.4.

Lemma 2.7. *For every open set $O\subset\mathcal{M}(\mathbb{R})\times\tilde{\mathcal{M}}_*(\mathbb{R}\times\mathbb{R})$,*

$$\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\Big\{(L^1,L^2)\in O\Big\}\geq -\inf_{(\omega,\varpi)\in O} I(\omega,\varpi).$$

Proof. Suppose $(\omega,\varpi)\in O$, with $\varpi\ll C\omega\otimes\omega$. Define $\tilde{f}_\omega\colon\mathbb{R}\to\mathbb{R}$ by

$$\tilde{f}_\omega(a) = \begin{cases} \log\frac{\omega[dx]}{\ell[dx]}, & \text{if } \omega[dx]>0,\\ 0, & \text{otherwise.}\end{cases}$$

and $\tilde{g}_\varpi\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$\tilde{g}_\varpi[x,y] = \begin{cases} \log\frac{\varpi[dx,dy]}{C[dx,dy]\omega[dx]\omega[dy]}, & \text{if } \varpi[dx,dy]>0,\\ 0, & \text{otherwise.}\end{cases}$$

In addition, we let $\tilde{\beta}_\varpi[x,y] = C[dx,dy](1-e^{\tilde{g}_\varpi[x,y]})$ and note that $\tilde{\beta}_\varpi[x,y] = \lim_{n\to\infty}\tilde{h}_{\varpi,n}[x,y]$, for all $a,b\in\mathbb{R}$ where

$$\tilde{h}_{\varpi,n}[x,y] = \log\Big[\big(1-p_n[dx,dy]+p_n[dx,dy]e^{\tilde{g}_\varpi[x,y]}\big)^{-n}\Big].$$

Choose B^1_ω, B^2_ϖ open neighbourhoods of ω,ϖ, such that $B^1_\omega\times B^2_\varpi\subset O$ and for all $(\tilde{\omega},\tilde{\varpi})\in B^1_\omega\times B^2_\varpi$

$$\langle\tilde{f}_\omega,\omega\rangle + \tfrac{1}{2}\,\langle\tilde{g}_\varpi,\varpi\rangle + \tfrac{1}{2}\,\langle\tilde{\beta}_\varpi,\omega\otimes\omega\rangle - \varepsilon \leq \langle\tilde{f}_\omega,\tilde{\omega}\rangle + \tfrac{1}{2}\,\langle\tilde{g}_\varpi,\tilde{\varpi}\rangle + \tfrac{1}{2}\,\langle\tilde{\beta}_\varpi,\tilde{\omega}\otimes\tilde{\omega}\rangle.$$

We now use $\tilde{\mathbb{P}}$, the probability measure obtained by transforming $\mathbb{P}$ using the functions $\tilde{f}_\omega$, $\tilde{g}_\varpi$. Note that the spin law in the transformed measure is now ω, and the connection probabilities $\tilde{p}_n[dx,dy]$ satisfy $n\,\tilde{p}_n[dx,dy] \to \varpi[dx,dy]/(\omega[dx]\omega[dy]) =: \tilde{C}[dx,dy]$, as $n\to\infty$. Using (2.4), we obtain

$$\begin{aligned}
\mathbb{P}\Big\{(L^1,L^2)\in O\Big\} &\geq \tilde{\mathbb{E}}\Big\{\tfrac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(X)\mathbb{1}_{\{(L^1,L^2)\in B^1_\omega\times B^2_\varpi\}}\Big\}\\
&= \tilde{\mathbb{E}}\Big\{\prod_{u\in V} e^{-\tilde{f}_\omega[s(\eta(u))]}\prod_{(u,v)\in E} e^{-\tilde{g}_\varpi[s(\eta(u)),s(\eta(v))]}\prod_{(u,v)\in\mathcal{E}} e^{-\frac{1}{n}\tilde{h}_{\varpi,n}[s(\eta(u)),s(\eta(v))]}\mathbb{1}_{\{(L^1,L^2)\in B^1_\omega\times B^2_\varpi\}}\Big\}\\
&= \tilde{\mathbb{E}}\Big\{e^{-n\langle L^1,\tilde{f}_\omega\rangle - n\frac{1}{2}\langle L^2,\tilde{g}_\varpi\rangle - n\frac{1}{2}\langle L^1\otimes L^1,\tilde{g}_\varpi\rangle+\frac{1}{2}\langle L^1_\Delta,\tilde{h}_{\varpi,n}\rangle}\times\mathbb{1}_{\{(L^1,L^2)\in B^1_\omega\times B^2_\varpi\}}\Big\}\\
&\geq \exp\big(-n\langle\tilde{f}_\omega,\omega\rangle - n\tfrac{1}{2}\langle\tilde{g}_\varpi,\varpi\rangle - n\tfrac{1}{2}\langle\tilde{\beta}_\varpi,\omega\otimes\omega\rangle + m - n\varepsilon\big)\times\tilde{\mathbb{P}}\Big\{(L^1,L^2)\in B^1_\omega\times B^2_\varpi\Big\},
\end{aligned}$$

where $m := 0\wedge\inf_{x\in\mathbb{R}}\tilde{\beta}[x,x]$. Therefore, by (2.5), we have

$$\begin{aligned}
&\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\Big\{(L^1,L^2)\in O\Big\}\\
&\geq -\langle\tilde{f}_\omega,\omega\rangle - \tfrac{1}{2}\langle\tilde{g}_\varpi,\varpi\rangle - \tfrac{1}{2}\langle\tilde{\beta}_\varpi,\omega\otimes\omega\rangle - \varepsilon + \liminf_{n\to\infty}\tfrac{1}{n}\log\tilde{\mathbb{P}}\Big\{(L^1,L^2)\in B^1_\omega\times B^2_\varpi\Big\}.
\end{aligned}$$

The result follows once we prove that

$$\liminf_{n\to\infty}\tfrac{1}{n}\log\tilde{\mathbb{P}}\Big\{(L^1,L^2)\in B^1_\omega\times B^2_\varpi\Big\} = 0. \tag{2.10}$$

We use the upper bound (but now with the law $\mathbb{P}$ replaced by $\tilde{\mathbb{P}}$) to prove (2.10). Then we obtain

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\tilde{\mathbb{P}}\big\{(L^1,L^2)\in(B^1_\omega\times B^2_\varpi)^c\big\}\leq -\inf_{(\tilde{\omega},\tilde{\varpi})\in\tilde{F}}\tilde{I}(\tilde{\omega},\tilde{\varpi}),$$

where $\tilde{F} = (B^1_\omega\times B^2_\varpi)^c$ and

$$\tilde{I}(\tilde{\omega},\tilde{\varpi}) := H(\tilde{\omega}\,\|\,\omega) + \tfrac{1}{2}H\big(\tilde{\varpi}\,\|\,\tilde{C}\tilde{\omega}\otimes\tilde{\omega}\big) + \tfrac{1}{2}\|\tilde{C}\tilde{\omega}\otimes\tilde{\omega}\| - \tfrac{1}{2}\|\tilde{\varpi}\|.$$

It therefore suffices to show that the infimum is positive. Suppose for contradiction that there exists a sequence $(\tilde{\omega}_n,\tilde{\varpi}_n)\in\tilde{F}$ with $\tilde{I}(\tilde{\omega}_n,\tilde{\varpi}_n)\downarrow 0$. Then, because $\tilde{I}$ is a good rate function and its level sets are compact, and by lower semicontinuity of the mapping $(\tilde{\omega},\tilde{\varpi})\mapsto\tilde{I}(\tilde{\omega},\tilde{\varpi})$, we can construct a limit point $(\tilde{\omega},\tilde{\varpi})\in\tilde{F}$ with $\tilde{I}(\tilde{\omega},\tilde{\varpi}) = 0$. By Lemma 2.6 this implies $H(\tilde{\omega}\,\|\,\omega) = 0$ and $\mathfrak{H}_C(\tilde{\varpi}\,\|\,\tilde{\omega}) = 0$, hence $\tilde{\omega} = \omega$, and $\tilde{\varpi} = \tilde{C}\tilde{\omega}\otimes\tilde{\omega} = \varpi$ contradicting $(\tilde{\omega},\tilde{\varpi})\in\tilde{F}$. ■

3. Results and Discussion.

3.1 Thermodynamic Limits

Theorem 3.1. *Suppose that X is a spinned random graph with bond probabilities $p_n\colon\mathcal{X}\times\mathcal{X}\to[0,1]$ satisfying $np_n[s(x),s(y)]\to C[s(x),s(y)]$, for all $x,y\in\{-1,1\}$ and some continuous symmetric function $C\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$ that satisfies* (2.1) *above. Then, the asymptotic pressure per particle is*

$$\phi(\beta,B) = \tfrac{1}{4}\Big\{(e^\beta-1)C[\sqrt{\beta}B(1),\sqrt{\beta}B(1)] + (e^{-\beta}-1)C[-\sqrt{\beta}B(-1),\sqrt{\beta}B(1)]\Big\} + \tfrac{(3B(1)-B(-1))}{4}$$

Corollary 3.2. *Suppose that X is a spinned random graph with bond probabilities $p_n\colon\mathcal{X}\times\mathcal{X}\to[0,1]$ satisfying $np_n[s(x),s(y)]\to C[s(x),s(y)]$, for all $x,y\in\{-1,1\}$ and some twice differential symmetric function $C\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$ that satisfies* (2.1) *above. Then, for all $\beta\geq 0$, each of the following statements holds a.s.:*

(i) **Asymptotic total magnetization per site** *is given by*

$$M(\beta,\, B)=\frac{1}{2}-\frac{(e^{\beta}C_{1,1}-e^{-\beta}C_{-1,1})+(e^{\beta}-1)\frac{dC_{1,1}}{d\beta}+(e^{-\beta}-1)\frac{dC_{-1,1}}{d\beta}}{e^{\beta}(C_{1,1}-C_{-1,-1})+(e^{\beta}-1)(\frac{dC_{1,1}}{d\beta}-\frac{dC_{-1,-1}}{d\beta})}$$

(ii) **Asymptotic internal energy** *is given by*

$$U(\beta,B)=-\frac{1}{4}\Big[C_{-1,-1}e^{\beta}-C_{-1,1}e^{-\beta}+(e^{-\beta}-1)\frac{dC_{-1,1}}{d\beta}+(e^{\beta}-1)\frac{dC_{-1,-1}}{d\beta}\Big]$$

(iii) **Asymptotic specific heat** *is given by*

$$\mathcal{H}(\beta,B)=\frac{\beta^2}{4}\big[C_{-1,-1}e^{\beta}+C_{-1,1}e^{\beta}+e^{\beta}\frac{dC_{-1,-1}}{d\beta}-2e^{-\beta}\frac{dC_{-1,1}}{d\beta}+(e^{-\beta}-1)\frac{dC_{-1,1}}{d\beta}+(e^{\beta}-1)\frac{dC_{-1,-1}}{d\beta}+e^{\beta}\frac{dC_{-1,-1}}{d\beta}\big]$$

Remark 1 :**Asymptotic susceptibility of the system** is given by

$$\mathcal{S}(\beta,B):=\frac{\partial^2\phi}{\partial B(-1)^2}+\frac{\partial^2\phi}{\partial B(1)^2}$$

Theorem 3.3. *Let β_c be the solution of equation $C_{-1,1}=\frac{1}{2}C_{1,1}+\frac{1}{2}C_{-1,-1}$. Then, $\beta=\beta_c$, separates the non magnetized phase (*diamagnetic state*) from the magnetized one (*paramagnetic state*).Further, for $B(1)+B(-1)=0$, the continuous function $B\to\phi(\beta,B)$ exhibits a discontinuous derivative.i.e. the susceptibility becomes infinite at $\beta=\beta_c$ in $B(1)+B(-1)=0$.*

Corollary 3.4. *Let λ be the asymptotic average number of connectivity of erdos-renyi graphs. Then, there exists $\beta_c(\lambda)$ such the ferromagnetic Ising model on the erdos-renyi graphs exhibit*

(i) **diamagnetic properties** *if* $0<\beta<\beta_c(\lambda)$
(ii) **paramagnetic properties** *if* $\beta_c(\lambda)<\beta\le\infty$
(iii) **zero-magnetization properties** *if* $\beta=0$ *or* $\beta=\beta_c(\lambda)$.

3.2 Proof of Thermodynamics Limits. We kick start the proof of our Thermodynamics limit results by stating an important Lemma (Varadhan's Lemma, see Dembo et al. [1998, Theorem 4.3.1]), which is a key step in establishing our first result Theorem 3.1, without proof.

Lemma 3.5 (Varadhan). *Suppose the functional M_n from the space of finite spinned graphs to $\mathcal{M}$ satisfies the LDP withh good rate function $I:\mathcal{M}\to[0,\infty]$ and let $\Psi:\mathcal{M}\to\mathbb{R}$ be any continuous function. Assume further the following moment condition for some $\lambda>1$,*

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}\big[e^{n\lambda\Psi(M_n[X])}\big]<\infty.$$

Then,

$$\lim_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}\big[e^{n\lambda\Psi(M_n[X])}\big]=\sup_{m\in\mathcal{M}}\big\{\Psi(m)-I(m)\big\}.$$

Next we provide annealed asymptotics of the random partition fuction for the ferromagnetic Ising model on spinned random graphs, as the graph size goes to infinity.

Lemma 3.6. *Suppose that X is a spinned random graph with spin law ℓ and link probabilities $p_n\colon\mathcal{X}\times\mathcal{X}\to[0,1]$ satisfying $np_n[s(a),s(b)]\to C[s(a),s(b)]$, for all $a,b\in\{-1,1\}$ and some continuous symmetric function $C\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$ with $\sup_{x,y\in\mathbb{R}}C[x,\,y]<\infty$ and satisfies* (2.1) *above. Then,*

$$\lim_{n\to\infty}\tfrac{1}{n}\log\mathsf{E}[Z_X(\beta,B)]=\log 2+\tfrac{1}{4}\Big\{(e^{\beta}-1)C[\sqrt{\beta}B(1),\sqrt{\beta}B(1)]+(e^{-\beta}-1)C[-\sqrt{\beta}B(-1),\sqrt{\beta}B(1)]\Big\}+\tfrac{(3B(1)-B(-1))}{4}$$

Proof. Recall the random partition function of the Ising model on the spinned graph from section 2 and write it as integral of some function with respect to our empirical measures:

$$\mathsf{E}[Z_X(\beta,B)] := 2^n\mathbb{E}\Big[\exp\Big\{\tfrac{n}{2}\int \tfrac{s(x)}{B(x)}\tfrac{s(x)}{B(y)}L^2[ds(x),ds(y)] + n\int \tfrac{s(x)}{\sqrt{\beta}}L^1[ds(x)]\Big\}\Big] \tag{3.1}$$

Now using the (Varadhan) Lemma 3.5 and Theorem 3.4 we obtain,

$$\begin{aligned}\lim_{n\to\infty}\tfrac{1}{n}\log\mathsf{E}[Z_X(\beta,B)] &= \log 2 + \sup\Big\{\tfrac{1}{2}\int \tfrac{s(x)}{B(x)}\tfrac{s(x)}{B(y)}\varpi[ds(x),\,ds(y)] + \int \tfrac{s(x)}{\sqrt{\beta}}\omega[ds(x)] - I(\omega,\varpi)\\ &\qquad : \omega\in\mathcal{X}(\{s(-1),s(1)\}), \varpi\in\mathcal{M}_*(\{s(-1),s(1)\}\times\{s(-1),s(1)\})\Big\}\\ &= \sup\Big\{\tfrac{\beta}{2}\big(\varpi(\Delta)-\varpi(\Delta^{c})\big) + B(1)x - B(-1)(1-x) - x\log(x) - (1-x)\log(1-x)\\ &\qquad - \tfrac{1}{2}\big(H(\varpi\,\|\,\omega_x) + C_{1,1}x + C_{-1,-1}(1-x) + 2C_{-1,1}x(1-x) - \|\varpi\|\big)\Big\},\end{aligned} \tag{3.2}$$

where Δ is the diagonal in $\{s(-1),s(1)\}\times\{s(-1),s(1)\}$, and the supremum is over all $x\in[0,1]$ and $\varpi\in\mathcal{M}_*(\{s(-1),s(1)\}\times\{s(-1),s(1)\})$, and the measure $\omega_x\in\tilde{\mathcal{M}}_*(\{s(-1),s(1)\}\times\{s(-1),s(1)\})$ is defined by

$$\omega_x[s(i),s(j)] = C_{i,j}x^{(2+i+j)/2}(1-x)^{(2-i-j)/2} \text{ for } i,j\in\{-1,1\}\,.$$

We take the partial derivatives of (3.2) with respect to ϖ, and set our results to zero to get,

$$\begin{aligned}\varpi[s(1),s(1)] &= e^{\beta}\omega_x[s(1),s(1)],\\ \varpi[s(-1),s(-1)] &= e^{\beta}\omega_x[s(-1),s(-1)]\\ \varpi[s(1),s(-1)] &= e^{-\beta}\omega_x[s(1),s(-1)]\\ \varpi[s(-1),s(1)] &= e^{-\beta}\omega_x[s(-1),s(1)].\end{aligned}$$

$$\begin{aligned}&\lim_{n\to\infty}\tfrac{1}{n}\log\mathsf{E}[Z_X(\beta,B)]\\ &= \sup_{x\in[0,1]}\Big\{\tfrac{\beta}{2}e^{\beta}\omega_x[s(1),s(1)] + \tfrac{\beta}{2}e^{\beta}\omega_x[s(-1),s(-1) - \beta e^{-\beta}\omega_x[s(-1),s(1)] + B(1)x - B(-1)(1-x)\\ &- x\log(x) - (1-x)\log(1-x) - \tfrac{\beta}{2}e^{\beta}\omega_x[s(1),s(1)] - \tfrac{\beta}{2}e^{\beta}\omega_x[s(-1),s(-1)] + \beta e^{-\beta}\omega_x[s(-1),s(1)] - \tfrac{1}{2}C_{1,1}x\\ &- \tfrac{1}{2}C_{-1,-1}(1-x) - C_{-1,1}x(1-x) + \tfrac{1}{2}e^{\beta}\omega_x[s(1),s(1)] + \tfrac{1}{2}e^{\beta}\omega_x[s(-1),s(-1)] + e^{-\beta}\omega_x[s(-1),s(1)]\Big\}\\ &= \sup_{x\in[0,1]}\Big\{-x\log(x) - (1-x)\log(1-x) - \tfrac{1}{2}C_{1,1}x^2 - \tfrac{1}{2}C_{-1,-1}(1-x)^2 + B(1)x - B(-1)(1-x)\\ &- C_{-1,1}x(1-x) + \tfrac{1}{2}e^{\beta}C_{1,1}x^2 + \tfrac{1}{2}e^{\beta}C_{-1,-1}(1-x)^2 + e^{-\beta}C_{-1,1}x(1-x)\Big\}\\ &= \sup_{x\in[0,1]}\Big\{\lambda_1(x)+\lambda_2(x)\Big\}\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\lambda_1(x) &= -x\log(x) - (1-x)\log(1-x)\\ \lambda_2(x) &= a_1x^2 + a_2(1-x)^2 + a_3x(1-x) + B(1)x - B(-1)(1-x)\end{aligned}$$

$a_1 = \frac{1}{2}\,C_{1,1}(e^{\beta}-1)$, $a_2 = \frac{1}{2}\,C_{-1,-1}(e^{\beta}-1)$ and $a_3 =\, C_{-1,1}(e^{-\beta}-1)$.

Elementary calculus shows that the grobal maxima of the functions λ_1 and λ_2 are attained at the value $x = \frac{1}{2}$, when $a_2 - a_1 = B(1) + B(-1)$. Therefore we have

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathsf{E}[Z_X(\beta, B)] = \log 2 + \tfrac{1}{4}(2a_1 + a_3) + \tfrac{1}{4}(3B(1) - B(-1))$$
$$= \log 2 + \tfrac{1}{4}\Big\{(e^{\beta} - 1)C[\sqrt{\beta}B(1), \sqrt{\beta}B(1)] + (e^{-\beta} - 1)C[-\sqrt{\beta}B(-1), \sqrt{\beta}B(1)]\Big\} + \tfrac{3B(1)-B(-1))}{4}$$

■

Lemma 3.7.

$$\lim_{n\to\infty} \frac{\Phi_n(\beta, B) - \log 2}{n^4} = \lim_{n\to\infty} \frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)/2^n]}{n^4} \quad \textit{with prob. } 1$$

Proof.

$$\mathbb{E}\Big|\frac{\Phi_n(\beta, B) - \log 2}{n^4} - \frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)/2^n]}{n^4}\Big| \le \tfrac{\beta}{4n^2} + \tfrac{B}{n^3} + \tfrac{\phi(\beta,B)-\log 2+\varepsilon}{n^4} \le \tfrac{A}{n^2}$$

for some $A = A(\beta, B, \varepsilon)$.

Now

$$\sum_{n=1}^{\infty} \mathbb{E}\Big|\frac{\Phi_n(\beta, B) - \log 2}{n^4} - \frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)/2^n]}{n^4}\Big| \le \sum_{n=1}^{\infty} \tfrac{A}{n^2} < \infty \tag{3.4}$$

Using the Markov's inequality and (3.7) we have that,

$$\sum_{n=1}^{\infty} \mathbb{P}\Big\{\Big|\frac{\Phi_n(\beta, B) - \log 2}{n^4} - \frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)/2^n]}{n^4}\Big| \ge \varepsilon\Big\} \le \tfrac{1}{\varepsilon}\sum_{n=1}^{\infty} \mathbb{E}\Big|\frac{\Phi_n(\beta, B) - \log 2}{n^4} - \frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)/2^n]}{n^4}\Big|$$
$$\le \tfrac{1}{\varepsilon}\sum_{n=1}^{\infty} \tfrac{A}{n^2} < \infty$$

By the Borel-Cantelli Lemma, this estimate result in

$$\lim_{n\to\infty} \frac{\Phi_n(\beta, B) - \log 2}{n^4} = \lim_{n\to\infty} \frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)/2^n]}{n^4} \text{ almost surely}$$

■

We use Lemmas (3.2) and (3.7), to obtain

$$\begin{aligned}
\lim_{n\to\infty} \Phi_n(\beta, B) = \lim_{n\to\infty} n^4 \frac{\Phi_n(\beta, B) - \log 2 + \log 2}{n^4} &= \lim_{n\to\infty} n^4 \lim_{n\to\infty} \frac{\Phi_n(\beta, B) - \log 2}{n^4} + \log 2 \\
&= \lim_{n\to\infty} n^4 \lim_{n\to\infty} \Big(\frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)/2^n]}{n^4}\Big) + \log 2 \\
&= \lim_{n\to\infty} n^4 \Big(\frac{\frac{1}{n}\log \mathsf{E}[Z_X(\beta, B)]}{n^4}\Big) \\
&= \lim_{n\to\infty} \tfrac{1}{n}\log \mathsf{E}[Z_X(\beta, B)]\big) \\
&= \phi(\beta, B)
\end{aligned}$$

with probability 1, which concludes the proof of our main Thermodynamic Limiting results.

Corollary 3.2

(i) We take the partial derivative of $\phi(\beta, B)$, using the chain rule of differentiation, with respect to $B(1)$ and $B(-1)$ to get

$$\frac{\partial\phi}{\partial B(1)} = \frac{1}{4}\big[(e^{\beta}C_{1,1} - e^{-\beta}C_{-1,1}) + (e^{\beta}-1)\tfrac{dC_{1,1}}{d\beta} + (e^{-\beta}-1)\tfrac{dC_{-1,1}}{d\beta}\big]\tfrac{\partial\beta}{\partial B(1)} - \frac{1}{4} \tag{3.5}$$

and

$$\frac{\partial\phi}{\partial B(-1)} = \frac{1}{4}\big[(e^{\beta}C_{1,1} - e^{-\beta}C_{-1,1}) + (e^{\beta}-1)\tfrac{dC_{1,1}}{d\beta} + (e^{-\beta}-1)\tfrac{dC_{-1,1}}{d\beta}\big]\tfrac{\partial\beta}{\partial B(-1)} + \frac{3}{4}. \tag{3.6}$$

Now the energetic preference condition gives

$$\tfrac{\partial\beta}{\partial B(1)} = \tfrac{\partial\beta}{\partial B(-1)} = -2\big[e^{\beta}(C_{1,1} - C_{-1,-1}) + (e^{\beta}-1)(\tfrac{dC_{1,1}}{d\beta} - \tfrac{dC_{-1,-1}}{d\beta})\big]^{-1}. \tag{3.7}$$

We write this expression $\frac{\partial\beta}{\partial B(1)}$ in (3.5) and (3.6), and adding them together we have the $M(\beta, B)$.

(ii) We take the partial derivative of $\phi(\beta, B)$, using the chain rule of differentiation, with respect to β to get

$$\frac{\partial\phi}{\partial\beta} = \frac{1}{4}\big[(e^{\beta}C_{1,1} - e^{-\beta}C_{-1,1}) + (e^{\beta}-1)\tfrac{dC_{1,1}}{d\beta} + (e^{-\beta}-1)\tfrac{dC_{-1,1}}{d\beta}\big]\tfrac{\partial\beta}{\partial B(-1)} + \frac{1}{4}(3\tfrac{dB(1)}{d\beta} - \tfrac{dB(-1)}{d\beta}). \tag{3.8}$$

Now writing (3.7) in (3.8) and negating our results we obtain $U(\beta, B)$.

(iii) Differentiating U with respect to β and write the expression in the relation

$$\mathcal{H}(\beta, B) = -\beta^2\frac{dU}{d\beta}$$

we have the asymptotic specific heat.

Proof of Theorem 3.3: Let β_c be the solution of equation $C_{-1,1} = \frac{1}{2}C_{1,1} + \frac{1}{2}C_{-1,-1}$. Then, it is not uneasy to see that $(M(\beta_c, B) = 0$. Further, $B(1) + B(-1) = 0$, implies $C_{1,1} = C_{-1,-1}$ by the energetic preference condition, and $\beta \to M(\beta, B)$, the derivative of ϕ with respect to β is discontinuous for all B. Since the susceptibility is the derivative of $M(\beta, B)$ with respect to β, we have

$$\lim_{\beta\to\beta_c}\frac{dM}{d\beta} = \infty.$$

Proof of Corollary 3.4: Let λ be the asymptotic average number of connectivity of erdos-renyi graphs. Then, we have $np_n[s(x), s(y)] \to \lambda$, for all $x, y\{-1, 1\}$. Now take $C_{1,1} = C_{-1,1} = C_{-1,-1}$, and note that $C_{-1,1} = \frac{1}{2}C_{1,1} + \frac{1}{2}C_{-1,-1}$, for all $\beta \geq 0$. Using Theorem 3.3 we can find $\beta_c(\lambda)$ with separates the paramagnetic state from the diamagnetic state which ends the proof of the corollary.

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